Asymptotically flat solutions to the Ernst equation with reflection symmetry

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It is shown that the class of asymptotically flat solutions to the axisymmetric and stationary vacuum Einstein equations with reflection symmetry of the metric is uniquely characterized by a simple relation for the Ernst potential $f^{(u)}$ on the upper part of the symmetry axis (ζ–axis): $f^{(u)}(\zeta)\bar{f}^{(u)}(-\zeta) = 1$. This result generalizes a well–known fact from potential theory: Axisymmetric solutions to the Laplace equation that vanish at infinity and have reflection symmetry with respect to the plane $\zeta = 0$ are characterized by a potential that is an odd function of ζ on the upper part of the ζ–axis.

I. INTRODUCTION

The general solution of the axisymmetric Laplace equation

$$\triangle U \equiv U,_{\rho\rho} + \frac{1}{\rho} U,_{\rho} + U,_{\zeta\zeta} = 0 \tag{1}$$

that is regular outside some finite region and vanishes at infinity may be written in form of a multipole expansion

$$U = \sum_{n=0}^{\infty} c_n \, r^{-(n+1)} \, \mathrm{P_n}(\cos\theta) \tag{2}$$

with

$$\rho = r\sin\theta, \quad \zeta = r\cos\theta. \tag{3}$$

Solutions with reflection symmetry

$$U(\rho, -\zeta) = U(\rho, \zeta) \tag{4}$$

are given by

$$c_n = 0 \quad \text{for} \quad n = 1, 3, 5, \ldots \tag{5}$$

This leads to the following form of the potential $U^{(u)}$ on the upper part of the axis ($\theta = 0$):

$$U^{(u)}(\zeta) = \sum_{k=0}^{\infty} c_{2k} \, \zeta^{-(2k+1)}. \tag{6}$$

Hence, solutions with reflection symmetry are uniquely characterized by an odd function $U^{(u)}(\zeta)$:

$$U^{(u)}(-\zeta) = -U^{(u)}(\zeta). \tag{7}$$

The aim of the present paper is to prove a relation that generalizes Eq. (7) to the case of solutions to the axisymmetric and stationary vacuum Einstein equations with reflection symmetry. As is well known these equations are equivalent to an equation for the complex Ernst potential $f(\rho, \zeta)$:

$$(\Re f)\triangle f = (\nabla f)^2. \tag{8}$$

Solutions with reflection symmetry of the metric are defined by

$$f(\rho, -\zeta) = \bar{f}(\rho, \zeta), \tag{9}$$

where a bar denotes complex conjugation. Eq. (9) is the analogue of Eq. (4). (It should be noted that the real part $\Re f$ of f is directly related to some metric coefficient, whereas its imaginary part is related to some other metric

coefficient in such a way, that symmetry of the metric means antisymmetry of $\Im f$.) The analogue of Eq. (7) will be shown to be

$$f^{(u)}(\zeta)\bar{f}^{(u)}(-\zeta) = 1. \tag{10}$$

The Laplace equation (1) is a special case of the Ernst equation (8) for real Ernst potentials

$$f = \exp(2U). \tag{11}$$

In this way it can easily be verified that Eqs. (4) and (7) are indeed special cases of (9) and (10), respectively.

Our prove of relation (10) is based upon the existence of a Linear Problem whose integrability condition is just the Ernst equation.

II. THE LINEAR PROBLEM RELATED TO THE ERNST EQUATION

Introducing complex variables

$$z = \rho + i\zeta, \quad \bar{z} = \rho - i\zeta \tag{12}$$

the Ernst equation takes the form

$$(f + \bar{f})\{(\rho f,_{\bar{z}}),_z + (\rho f,_z),_{\bar{z}}\} = 4\rho f,_z f,_{\bar{z}} . \tag{13}$$

Eq. (13) is the integrability condition of a related Linear Problem for the 2×2–matrix function $\Phi(\lambda, z, \bar{z})$:

$$\Phi,_z = \left\{ \begin{pmatrix} N & 0 \\ 0 & M \end{pmatrix} + \lambda \begin{pmatrix} 0 & N \\ M & 0 \end{pmatrix} \right\} \Phi, \tag{14}$$

$$\Phi,_{\bar{z}} = \left\{ \begin{pmatrix} \bar{M} & 0 \\ 0 & \bar{N} \end{pmatrix} + \frac{1}{\lambda} \begin{pmatrix} 0 & \bar{M} \\ \bar{N} & 0 \end{pmatrix} \right\} \Phi, \tag{15}$$

cf. [1]. M and N depend on z and $\bar{z}$ but not on λ, which is defined as

$$\lambda = \sqrt{\frac{K - i\bar{z}}{K + iz}}, \tag{16}$$

where K is an additional complex variable called the spectral parameter[1]. Consequently,

$$\lambda,_z = \frac{\lambda}{4\rho}(\lambda^2 - 1), \quad \lambda,_{\bar{z}} = \frac{1}{4\rho\lambda}(\lambda^2 - 1). \tag{17}$$

The condition $\Phi,_{z\bar{z}} = \Phi,_{\bar{z}z}$ then implies a first order system of nonlinear equations for the functions M and N which is equivalent to the Ernst equation via

$$M = \frac{f,_z}{f + \bar{f}}, \quad N = \frac{\bar{f},_z}{f + \bar{f}}. \tag{18}$$

Without loss of generality the following structure of the matrix Φ may be assumed:

$$\Phi = \begin{pmatrix} \psi(\lambda, z, \bar{z}) & \psi(-\lambda, z, \bar{z}) \\ \chi(\lambda, z, \bar{z}) & -\chi(-\lambda, z, \bar{z}) \end{pmatrix}, \tag{19}$$

together with

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[1] For some considerations of the matrix function Φ it turns out to be appropriate to switch to a different set of independent variables, e. g. K, ρ, ζ instead of $\lambda, z, \bar{z}$.

$$\overline{\psi(\frac{1}{\lambda}, z, \bar{z})} = \chi(\lambda, z, \bar{z}). \tag{20}$$

For $K \to \infty$ and $\lambda = -1$ the functions ψ and χ may be normalized by

$$\psi(-1, z, \bar{z}) = \chi(-1, z, \bar{z}) = 1. \tag{21}$$

The related solution to the Ernst equation is given by

$$f(\rho, \zeta) \equiv \chi(1, z, \bar{z}) \quad (K \to \infty). \tag{22}$$

We mention that, according to (16), the complex λ–plane is related to a two–sheeted Riemann K–surface, with the branching points $K = \mathrm{i}\bar{z}$ and $K = -\mathrm{i}z$.

III. SOLUTION OF THE LINEAR PROBLEM ON THE AXIS

We consider solutions of the Ernst equation that are regular outside some finite region. Therefore, on the axis $\rho = 0$, we distinguish two regions: an upper part $\zeta > \zeta_+$ and a lower part $\zeta < \zeta_-$, where we assume the solution to be regular. We introduce the following notation:

$$f^{(u)}(\zeta) \equiv f(0, \zeta), \quad \text{with} \quad \zeta > \zeta_+; \quad f^{(l)}(\zeta) \equiv f(0, \zeta), \quad \text{with} \quad \zeta < \zeta_-. \tag{23}$$

For $\rho = 0$ and $K \neq \zeta$ Eq. (16) leads to

$$\lambda = \pm 1. \tag{24}$$

This allows us to integrate the Linear Problem (14), (15) along the axis[2]:

$$\begin{pmatrix} \psi \\ \chi \end{pmatrix} = F_{\pm}^{(u/l)}(K) \begin{pmatrix} \bar{f}^{(u/l)}(\zeta) \\ \pm f^{(u/l)}(\zeta) \end{pmatrix} + G_{\pm}^{(u/l)}(K) \begin{pmatrix} 1 \\ \mp 1 \end{pmatrix}. \tag{25}$$

Two of these 8 integration 'constants' $F_{\pm}^{(u/l)}(K)$ and $G_{\pm}^{(u/l)}(K)$ (they may depend on K, but not on ζ) can be fixed by prescribing ψ and χ at one point $\zeta = \zeta_0$ on the axis, for K–values from one of the two sheets of the Riemann surface ($\lambda = +1$ or $\lambda = -1$). We choose some $\zeta_0 > \zeta_+$ and assume for $\lambda = -1$

$$\rho = 0, \zeta = \zeta_0, \lambda = -1: \quad \psi \equiv 1, \quad \chi \equiv 1. \tag{26}$$

This choice is consistent with Eqs. (20) and (21). From Eqs. (25) and (26) we conclude

$$F_{-}^{(u)}(K) \equiv 0, \quad G_{-}^{(u)}(K) \equiv 1. \tag{27}$$

Now we use the assumption of asymptotic flatness, i. e.

$$f \to 1 \quad \text{as} \quad \rho^2 + \zeta^2 \to \infty. \tag{28}$$

As a consequence, the coefficients M and N in the Linear Problem (14), (15) vanish asymptotically such that ψ and χ do not change on the half–circle in the ρ–ζ–plane

$$\rho = R \sin\theta, \quad \zeta = R\cos\theta, \quad 0 \le \theta \le \pi, \tag{29}$$

in the limit $R \to \infty$. In this limit we find from (16)

$$\lambda = \pm \exp(\mathrm{i}\theta), \tag{30}$$

[2]Note that, for any function φ depending on $\lambda, z, \bar{z}$, $\lim_{\rho\to 0} \varphi(\lambda, z, \bar{z})$ is a function of the *two* variables K and ζ, in general. For example, $\lim_{\rho\to 0}(1-\lambda^2)/(z+\bar{z}) = \lim_{\rho\to 0} i/(K+iz) = i/(K-\zeta)$.

i. e. λ changes from ± 1 to ∓ 1 as θ changes from 0 to π. Therefore, we may conclude from Eqs. (25) for $\zeta \to \pm\infty$

$$F_\pm^{(u)} + G_\pm^{(u)} = F_\mp^{(l)} + G_\mp^{(l)}, \quad F_\pm^{(u)} - G_\pm^{(u)} = -F_\mp^{(l)} + G_\mp^{(l)}, \tag{31}$$

i. e.

$$F_\pm^{(u)} = G_\mp^{(l)}, \quad G_\pm^{(u)} = F_\mp^{(l)}. \tag{32}$$

This means, together with Eq. (27), that all these functions may be expressed in terms of two of them, say $F_+^{(u)}$ and $G_+^{(u)}$. With the new notation

$$F(K) \equiv F_+^{(u)}(K), \quad G(K) \equiv G_+^{(u)}(K) \tag{33}$$

we may summarize these findings as follows

$$\begin{array}{lll} \underline{\zeta > \zeta_+:} & \lambda = -1: & \psi = \chi = 1, \\ & \lambda = +1: & \psi = F(K)\bar{f}^{(u)}(\zeta) + G(K), \quad \chi = F(K) f^{(u)}(\zeta) - G(K), \\ \underline{\zeta < \zeta_-:} & \lambda = +1: & \psi = \bar{f}^{(l)}(\zeta), \quad \chi = f^{(l)}(\zeta), \\ & \lambda = -1: & \psi = G(K)\bar{f}^{(l)}(\zeta) + F(K), \quad \chi = -G(K) f^{(l)}(\zeta) + F(K). \end{array} \tag{34}$$

The functions $F(K)$ and $G(K)$ characterize a solution $f(\rho,\zeta)$ of the Ernst equation uniquely. They are directly related to the Ernst potential on the axis. This can be seen by taking $K = \zeta$, i. e. by going to the branching points of Eq. (16). (Note that $K = \mathrm{i}\bar{z}$ and $K = -\mathrm{i}z$ coincide for $\rho = 0$.) For $K = \zeta$ the values of ψ and χ must be unique, i. e. the formulae for $\lambda = +1$ and $\lambda = -1$ in Eqs. (34) have to be identified. This leads to

$$f^{(u)}(\zeta) = \frac{1 + G(\zeta)}{F(\zeta)}, \quad \bar{f}^{(u)}(\zeta) = \frac{1 - G(\zeta)}{F(\zeta)}, \quad f^{(l)}(\zeta) = \frac{F(\zeta)}{1 + G(\zeta)}, \quad \bar{f}^{(l)}(\zeta) = \frac{F(\zeta)}{1 - G(\zeta)}. \tag{35}$$

On the other hand, we obtain

$$F(\zeta) = \frac{1}{\Re f^{(u)}(\zeta)}, \quad G(\zeta) = \mathrm{i}\,\frac{\Im f^{(u)}(\zeta)}{\Re f^{(u)}(\zeta)}. \tag{36}$$

We note that the function $F(K)$ is related to the determinant of $\Phi(\lambda, z, \bar{z})$. From Eqs. (14), (15) and (18) one finds that $(\Re f)^{-1} \det \Phi(\lambda, z, \bar{z})$ does not depend on z and $\bar{z}$. Calculating $\det \Phi$ for $\rho = 0, \zeta \to \infty$ then leads to the result

$$F(K) = -\frac{1}{f + \bar{f}} \det \Phi(\lambda, z, \bar{z}). \tag{37}$$

IV. SOLUTIONS WITH REFLECTION SYMMETRY

The symmetry relation (9) reads on the axis

$$f^{(l)}(-\zeta) = \bar{f}^{(u)}(\zeta) \quad (\zeta > \zeta_+ > 0). \tag{38}$$

(Because of the symmetry we may assume $\zeta_- = -\zeta_+$.) Then we find from Eqs. (35)

$$G(-\zeta) = G(\zeta), \quad G^2(\zeta) = 1 - F(\zeta)F(-\zeta). \tag{39}$$

These relations together with (36) lead to

$$\Re f^{(u)}(\zeta)\,\Im f^{(u)}(-\zeta) = \Re f^{(u)}(-\zeta)\,\Im f^{(u)}(\zeta), \quad \Re f^{(u)}(\zeta)\,\Re f^{(u)}(-\zeta) + \Im f^{(u)}(\zeta)\,\Im f^{(u)}(-\zeta) = 1. \tag{40}$$

This is equivalent to relation (10) that was to be shown. If, on the other hand, Eq. (10) is satisfied, one can easily invert the above conclusions to show Eq. (38), and from this, by the symmetric structure of the Ernst equation itself, the symmetry (9) in the whole space may be concluded. Hence we are lead to the following statement:

An asymptotically flat solution to the Ernst equation has reflection symmetry (9) if and only if the relation (10) is satisfied.

It is interesting to note that the reflection symmetry (9) of the Ernst potential leads to the following global property of $\Phi(\lambda, z, \bar{z})$:

$$\frac{1}{f+\bar{f}}\,\Phi^T(\lambda, z, \bar{z})\,\Phi(-\frac{1}{\lambda}, \bar{z}, z) = \begin{pmatrix} 1 & G(K) \\ G(K) & 1 \end{pmatrix} \tag{41}$$

together with

$$G(-K) = G(K), \quad G^2(K) = 1 - F(K)F(-K). \tag{42}$$

$F(K)$ and $G(K)$ are analytic functions, i. e. they may be obtained as analytic continuations of $F(\zeta)$ and $G(\zeta)$ defined by means of the axis potential according to (36).

V. DISCUSSION

Within the Newtonian theory of gravitation all equilibrium states of isolated self–gravitating fluids (i. e. stellar models) must have reflection symmetry through a plane (the 'equatorial plane') which is perpendicular to the rotation axis of the star, see [3]. It has been conjectured that stationary general relativistic stellar models must have reflection symmetry as well [4]. Therefore, reflection symmetric solutions to the Ernst equation describing the exterior gravitational fields of rotating bodies are of particular interest. Also the Kerr solution describing a rotating black hole has reflection symmetry. Accordingly, the Ernst potential of the Kerr solution satisfies (10). The same holds true for the solution describing a rigidly rotating disk of dust [5], [6]. A consequence of relation (10) is, that – as in the case of the Laplace equation – every second multipole moment vanishes. The non–vanishing moments are $M_0, M_2, M_4, \ldots$ for the 'mass moments' and $J_1, J_3, J_5, \ldots$ for the 'rotational moments'. (For a definition of these moments see [7].) This simplifies the investigation of reflection symmetric solutions.

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[1] G. Neugebauer, J. Phys. **A12**, L67 (1979) , **A13**, 1737 (1980).
[2] P. Kordas, to be published.
[3] L. Lichtenstein, *Gleichgewichtsfiguren rotierender Flüssigkeiten*, Springer Verlag Berlin 1933.
[4] L. Lindblom, Phil. Trans. R. Soc. Lond. **A340**, 353 (1992).
[5] G. Neugebauer and R. Meinel, Astrophys. J. **414**, L97 (1993).
[6] G. Neugebauer and R. Meinel , Phys. Rev. Lett. **73**, 2166 (1994).
[7] R. O. Hansen, J. Math. Phys. **15**, 46 (1974).